A NEW FAMILY OF TOPOLOGICAL RINGS WITH APPLICATIONS IN LINEAR SYSTEM THEORY

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Contents

Abstract. Motivated by the Schwartz space of tempered distributions $\mathscr{S}'$ and the Kondratiev space of stochastic distributions $\mathcal{S}_{-1}$ we define a wide family of nuclear spaces which are increasing unions of (duals of) Hilbert spaces $\mathscr{H}'_p, p \in \mathbb{N}$, with decreasing norms $\|\cdot\|_p$. The elements of these spaces are functions on a free commutative monoid. We characterize those rings in this family which satisfy an inequality of the form $\|f \diamond g\|_p \le A(p-q)\|f\|_q\|g\|_p$ for all $p \ge q+d$, where $\diamond$ denotes the convolution in the monoid, $A(p-q)$ is a strictly positive number and d is a fixed natural number (in this case we obtain commutative topological rings). Such an inequality holds in $\mathcal{S}_{-1}$, but not in $\mathscr{S}'$. We give an example of such a ring which contains $\mathscr{S}'$. We characterize invertible elements in these rings and present applications to linear system theory.

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1. Introduction

As is well known, the Schwartz space $\mathscr{S}'$ of complex tempered distributions can be viewed as the space of sequences of complex numbers $f=(f_n)_{n\in\mathbb{N}_0}$ subject to

$$\sum_{n\in\mathbb{N}_0}(n+1)^{-2p}|f_n|^2<\infty \text{ for some } p\in\mathbb{N}.$$

Setting

$$\|f\|_p=\left(\sum_{n\in\mathbb{N}_0}(n+1)^{-2p}|f_n|^2\right)^{1/2}<\infty,$$

one can represent $\mathscr{S}'$ as a union of an increasing sequence of Hilbert spaces $\mathscr{H}_1',\mathscr{H}_2',\ldots$of complex sequences, with decreasing norms:

$$\mathscr{H}_p'=\{f=(f_n)_{n\in\mathbb{N}_0}:\ \|f\|_p<\infty\}. \tag{1.1}$$

In Hida's white noise space theory, a counterpart of $\mathscr{S}'$ was introduced by Kondratiev, see [18] and the references therein, in the following way. We begin with a definition:

Definition 1.1. *ℓ denotes the set of sequences of elements of $\mathbb{N}_0$, indexed by $\mathbb{N}$,*

$$(\alpha_1,\alpha_2,\ldots)$$

where $\alpha_j\neq 0$ for at most a finite number of indices.

The stochastic counterpart of $\mathscr{S}'$ is the space $\mathcal{S}_{-1}$ of families of complex numbers $f=(f_\alpha)_{\alpha\in\ell}$ indexed by ℓ and such that

$$\sum_{\alpha\in\ell}|f_\alpha|^2(2\mathbb{N})^{-\alpha p}<\infty \text{ for some } p\in\mathbb{N}.$$

In the above expression, one sets

$$(2\mathbb{N})^\alpha=2^{\alpha_1}4^{\alpha_2}6^{\alpha_3}\cdots$$

We here set

$$\|f\|_p=\|(f_\alpha)_{\alpha\in\ell}\|_p=\left(\sum_{\alpha\in\ell}|f_\alpha|^2(2\mathbb{N})^{-\alpha p}\right)^{1/2}, \tag{1.2}$$

and denote

$$\mathscr{K}_p'=\{f=(f_\alpha)_{\alpha\in\ell}:\ \|f\|_p<\infty\}.$$

In a way similar to $\mathscr{S}'$, the space $\mathcal{S}_{-1}$ is the union of the increasing sequence of Hilbert spaces $\mathscr{K}_1',\mathscr{K}_2',\ldots$ with decreasing norms (1.2). The elements of $\mathcal{S}_{-1}$ are called stochastic distributions and play an important role in stochastic partial differential equations, see [18]. We also refer to the papers [2, 3, 4] where $\mathcal{S}_{-1}$ is used to develop a new

approach to linear stochastic systems. Recall that the convolution of two elements of $\mathcal{S}_{-1}$, $f = (f_\alpha)_{\alpha\in\ell}$ and $g = (g_\alpha)_{\alpha\in\ell}$, (which is called the Wick product and denoted by $\diamond$) is defined by

$$f \diamond g = \left(\sum_{\beta\leq\alpha} f_\beta g_{\alpha-\beta}\right)_{\alpha\in\ell}$$

and satisfies the inequality

$$\|f \diamond g\|_p \leq A(p-q)\|f\|_q\|g\|_p, \quad \text{for all } p \geq q+2, \tag{1.3}$$

where

$$A(p-q) = \left(\sum_{\alpha\in\ell}(2\mathbb{N})^{-(p-q)\alpha}\right)^{\frac{1}{2}}$$

is finite. This inequality is due to Våge, see [29], [18]. It expresses in particular that the multiplication operator

$$g \mapsto f \diamond g$$

is bounded from the Hilbert space $\mathscr{K}_p'$ into itself where $f \in \mathscr{K}_q'$ and $p \geq q+2$. It plays a key role in the applications mentioned above.

In view of (1.3) it is a natural question to ask if a similar inequality holds in $\mathscr{S}'$. Here lies an important structural difference between $\mathscr{S}'$ and $\mathcal{S}_{-1}$. If $f = (f_n)_{n\in\mathbb{N}_0}$ and $g = (g_n)_{n\in\mathbb{N}_0}$ belong to $\mathscr{S}'$, their convolution which is defined by

$$f * g = \left(\sum_{m\leq n} f_m g_{n-m}\right)_{n\in\mathbb{N}_0}$$

also belongs to $\mathscr{S}'$. Nevertheless, as will be proved in the sequel (see Corollary 6.1), one cannot have an inequality of the kind (1.3), that is:

$$\|f * g\|_p \leq B(p-q)\|f\|_q\|g\|_p, \quad \text{for all } p \geq q+d, \tag{1.4}$$

where $d \in \mathbb{N}$ is preassigned, $B(p-q) > 0$ is a constant which depends only on $p-q$ in $\mathbb{N}$, and where f runs through $\mathscr{H}_q'$ and g runs through $\mathscr{H}_p'$. Since such an inequality does not hold, the origin of the present study was to find nuclear spaces containing $\mathscr{S}'$ such that an appropriate inequality of the type (1.3) holds for the convolution.

More generally, in the present paper we define a wide family of nuclear spaces in terms of families of positive numbers, and give a characterization of those in which an inequality of the type (1.3) holds. Since such an inequality was first proved by Våge (in the setting of the Kondratiev space of stochastic distributions) we call these spaces Våge spaces. We

show that these spaces are in particular topological rings, and give a characterization of their invertible elements. We then consider the tensor product of two Våge spaces, and show that it is a Våge space too.

The Schwartz space of tempered distributions $\mathscr{S}'$ is not a Våge space. We define a Våge space, containing $\mathscr{S}'$. This space is the dual of a space which is included in the Schwartz space of test functions $\mathscr{S}$, consists of entire functions, and is invariant under the Fourier transform. One can thus define the Fourier transform on its dual, and study, as suggested to us by Palle Jorgensen, connections with the theory of hyperfunctions (see [8] for the latter). This will be done in another publication. We present some important properties of this space, and characterize it both in terms of sequences and in terms of entire functions.

The paper consists of seven sections besides the introduction, and we now describe its content. A family of spaces of sequences which includes the space $\mathscr{S}'$, and which we call *regular admissible spaces*, is introduced in Section 2. In Section 3 we characterize regular admissible spaces in which convolution satisfies an inequality of the type (1.3). We also prove that these spaces are topological rings, which we call Våge spaces. Invertible elements in these rings are characterized in Section 4. In Section 5 we prove that the tensor product of two Våge spaces is a Våge space. The last three sections are devoted to examples and applications. In Section 6 we define a Våge space which contains $\mathscr{S}'$. Some results of E. Hille on Hermite series play an important role in the arguments. In Section 7 we consider the Kondratiev space. Finally, applications to linear system theory are outlined in Section 8.

2. A new family of nuclear spaces and Gelfand triples

In this section we introduce a family of nuclear spaces of sequences which we use in the sequel. We begin with a definition and a preliminary result on sequences of numbers. Let A be a subset of $\mathbb{N}$. We denote

(2.1)
$$\ell_A = \left\{ \sum_{n\in A} \alpha_n e_n : \alpha_n \in \mathbb{N}_0,\ \alpha_n \neq 0 \text{ for at most finitely many } n \right\},$$

where, for $n \in \mathbb{N}$, we have denoted by e_n the sequence with all elements equal to 0, at exception of the n-th one, equal to 1. For two elements

$$\alpha = \sum_{n \in A} \alpha_n e_n \quad \text{and} \quad \beta = \sum_{n \in A} \beta_n e_n$$

in ℓ_A, we define $\alpha + \beta = \sum_{n \in A}(\alpha_n + \beta_n)e_n$. In other words, $(\ell_A, +, 0)$ is the free commutative monoid generated by the countable (or finite) set $\{e_n\}_{n \in A}$. Moreover, we consider the following partial order induced by the addition above: For $\alpha, \beta \in \ell_A$, we define $\alpha \leq \beta$ if there exists $\gamma \in \ell_A$ such that $\alpha + \gamma = \beta$.

Definition 2.1. *Let A be a subset of $\mathbb{N}$, and let ℓ_A be defined by* (2.1). *The family of strictly positive numbers $(a_\alpha)_{\alpha \in \ell_A}$ is admissible if $a_0 = 1$ and $a_{e_n} > 1$ for all $n \in A$.*
Let $d \in \mathbb{N}$. The admissible sequence is called d-regular (or simply regular) if furthermore

$$\sum_{n \in A} \frac{1}{a_{e_n}^d - 1} < \infty. \tag{2.2}$$

It is superexponential (resp. exponential) if

$$a_\alpha a_\beta \leq a_{\alpha+\beta} \quad (\textit{resp. } a_\alpha a_\beta = a_{\alpha+\beta}) \quad \forall \alpha, \beta \in \ell_A. \tag{2.3}$$

Two examples of exponential regular admissible families of positive numbers are as follows:

(a) The set A has cardinal one. Then, $\ell_A = \mathbb{N}_0$. We take $a_n = a^n$ with $a > 1$.

(b) We set $A = \mathbb{N}$. Then, $\ell_A = \ell$, where ℓ is as in Definition 1.1. We take

$$a_\alpha = (2\mathbb{N})^\alpha = 2^{\alpha_1} 4^{\alpha_2} 6^{\alpha_3} \cdots, \quad \alpha \in \ell. \tag{2.4}$$

As mentioned in the introduction, this last example occurs in Hida's white noise space theory, in the definition of spaces of stochastic distributions.

Proposition 2.2. *Let $(a_\alpha)_{\alpha \in \ell_A}$ be a superexponential d-regular admissible family of strictly positive numbers. Then,*

$$\sum_{\alpha \in \ell_A} a_\alpha^{-d} < \infty.$$

Furthermore, if $(a_\alpha)_{\alpha \in \ell_A}$ is exponential rather than superexponential then d-regularity is also necessary for the family $(a_\alpha^{-d})_{\alpha \in \ell_A}$ to be summable.

Proof. We first prove the theorem for the case $d=1$. It follows from (2.3) that for every $\alpha \in \ell_A$,

$$\prod_{n\in A} a_{e_n}^{\alpha_n} \leq a_\alpha.$$

Therefore,

$$\begin{aligned}\sum_{\alpha\in\ell_A} a_\alpha^{-1} &\leq \sum_{\alpha\in\ell_A}\prod_{n\in A} a_{e_n}^{-\alpha_n}\\ &= \prod_{n\in A}\sum_{\alpha_n=0}^{\infty} a_{e_n}^{-\alpha_n}\\ &= \prod_{n\in A}\frac{1}{1-a_{e_n}^{-1}}\\ &= \prod_{n\in A}\left(1+\frac{1}{a_{e_n}-1}\right) < \infty\end{aligned}$$

If $a_\alpha a_\beta = a_{\alpha+\beta}$, then $\forall\alpha\in\ell_A$, $\prod_{n\in A} a_{e_n}^{\alpha_n} = a_\alpha$. Therefore,

$$\sum_{\alpha\in A} a_\alpha^{-1} = \prod_{n\in A}\left(1+\frac{1}{a_{e_n}-1}\right),$$

which converges if and only if $\sum_{n\in A}\frac{1}{a_{e_n}-1} < \infty$. In case $d>1$, we take a_α^d instead of a_α. □

When $A=\mathbb{N}$ and a_α is given by (2.4) we obtain as a corollary a result of Zhang, proved in 1992, see [31], [18]. Zhang's proof uses Abel's convergence test. The result itself is necessary in order to present some important properties of Kondratiev spaces of stochastic test functions and stochastic distributions.

Corollary 2.3 (Zhang [31])**.** *Let $d\in\mathbb{N}$. $\sum_{\alpha\in\ell}(2\mathbb{N})^{-d\alpha} < \infty$ if and only if $d>1$.*

Proof. We take $\ell_A=\ell$. Thus

$$\sum_{\alpha\in\ell}(2\mathbb{N})^{-d\alpha} < \infty \quad\Longleftrightarrow\quad \sum_{n\in\mathbb{N}}\frac{1}{(2n)^d-1} < \infty,$$

which is true if and only if $d>1$. □

The spaces $\mathscr{S}'$ and $\mathcal{S}_{-1}$ are strong dual of Fréchet spaces. Namely, $\mathscr{S}'$ is the strong dual of the Schwartz space $\mathscr{S}$ of Hermite series

$$f(x)=\sum_{n=0}^{\infty} f_n\xi_n(x), \tag{2.5}$$

where $(\xi_n)_{n\in\mathbb{N}_0}$ denote the Hermite functions, and the coefficients f_n are complex numbers, and such that

$$\sum_{n\in\mathbb{N}_0}(n+1)^{2p}|f_n|^2<\infty \text{ for all } p\in\mathbb{N},$$

and $\mathcal{S}_{-1}$ is the dual of the Kondratiev space $\mathcal{S}_1$ of stochastic test functions which can be seen as families of complex numbers $(f_\alpha)_{\alpha\in\ell}$ indexed by ℓ and such that

$$\sum_{\alpha\in\ell}(\alpha!)^2(2\mathbb{N})^{\alpha p}|f_\alpha|^2<\infty \text{ for all } p\in\mathbb{N}.$$

The triples $(\mathscr{S},\mathbf{L}_2(\mathbb{R},dx),\mathscr{S}')$ and $(\mathcal{S}_1,\mathcal{W},\mathcal{S}_{-1})$, where $\mathcal{W}$ denotes the white noise space, are Gelfand triples. We now define a wide family of nuclear topological vector spaces, which includes $\mathscr{S}'$ and $\mathcal{S}_{-1}$, and which is closed under tensor products, as dual of certain Fréchet spaces. These spaces are built from two families of strictly positive numbers $(a_\alpha)_{\alpha\in\ell_A}$ and $(b_\alpha)_{\alpha\in\ell_A}$. We define the Hilbert space of sequences $(\xi_\alpha)_{\alpha\in\ell_A}$ of complex numbers

$$\mathcal{H}_b=\left\{(\xi_\alpha)_{\alpha\in\ell_A}:\sum_{\alpha\in\ell_A}|\xi_\alpha|^2b_\alpha<\infty\right\}, \tag{2.6}$$

and the countably normed space

$$\mathscr{F}_{a,b}=\left\{(\varphi_\alpha)_{\alpha\in\ell_A}:\sum_{\alpha\in\ell_A}|\varphi_\alpha|^2b_\alpha^2a_\alpha^p<\infty \text{ for all } p\in\mathbb{N}\right\}. \tag{2.7}$$

Theorem 2.4.

(a) *The space $\mathscr{F}_{a,b}$ endowed with the topology defined by the norms*

$$\|(\varphi_\alpha)_{\alpha\in\ell_A}\|^2=\sum_{\alpha\in\ell_A}|\varphi_\alpha|^2b_\alpha^2a_\alpha^p,\quad p=1,2,\ldots$$

is a Fréchet space.

(b) *Assume that for all $p\in\mathbb{N}$, $a_\alpha^pb_\alpha\geq 1$, for all but finitely many α. Then, $\mathcal{F}_{a,b}$ is continuously included in $\mathcal{H}_b$.*

(c) *The space $\mathscr{F}_{a,b}$ is nuclear if and only if for every $p\in\mathbb{N}$ there exists $q>p$ such that*

$$\sum_{\alpha\in\ell_A}a_\alpha^{-(q-p)}<\infty \tag{2.8}$$

Proof.

(a) For $p \in \mathbb{N}$ we set $\mathscr{F}_{a,b,p}$ to be the set of sequences $\xi = (\xi_\alpha)_{\alpha\in\ell_A}$ such that
$$\|\xi\|^2_{\mathscr{F}_{a,b,p}} \overset{\text{def.}}{=} \sum_{\alpha\in\ell_A} |\xi_\alpha|^2 b_\alpha^2 a_\alpha^p < \infty$$
Thus $\mathscr{F}_{a,b} = \bigcap_{p=1}^\infty \mathscr{F}_{a,b,p}$.

We first note that the norms are non decreasing and compatible (i.e. every sequence which is a Cauchy sequence with respect to the two norms and converges to zero with respect to one of them, converges to zero also with respect to the second).

(b) We have
$$\|(\xi_\alpha)_{\alpha\in\ell_A}\|^2_{\mathcal{H}_b} = \sum_{\alpha\in\ell_A} |\xi_\alpha|^2 b_\alpha \leq \sum_{\alpha\in\ell_A} |\xi_\alpha|^2 b_\alpha^2 a_\alpha^p = \|(\xi_\alpha)_{\alpha\in\ell_A}\|^2_{\mathscr{F}_{a,b,p}}.$$

(c) Defining $\delta_\alpha = (\delta_{\alpha,\beta})_{\beta\in\ell_A}$ such that $\delta_{\alpha,\beta} = 0$ if $\alpha \neq \beta$ and $\delta_{\alpha,\beta} = 1$ if $\alpha = \beta$, it is clear that $(\delta_\alpha a_\alpha^{-p/2} b_\alpha)_{\alpha\in\ell_A}$ is an orthonormal base for $\mathscr{F}_{a,b,p}$. Now, $\mathscr{F}_{a,b} = \bigcap_{p\in\mathbb{N}} \mathscr{F}_{a,b,p}$ is nuclear if and only if for every $p \in \mathbb{N}$ there exists $q > p$ such that the natural embedding $\iota : \mathscr{F}_{a,b,q} \to \mathscr{F}_{a,b,p}$ is Hilbert-Schmidt. The equality
$$\begin{aligned}
\text{tr }(\iota^*\iota) &= \sum_{\alpha\in\ell_A} \langle \iota^*\iota(\delta_\alpha a_\alpha^{-q/2} b_\alpha), (\delta_\alpha a_\alpha^{-q/2} b_\alpha)\rangle_q \\
&= \sum_{\alpha\in\ell_A} \|\iota(\delta_\alpha a_\alpha^{-q/2} b_\alpha)\|_p^2 \\
&= \sum_{\alpha\in\ell_A} a_\alpha^{-(q-p)} \|\iota(\delta_\alpha a_\alpha^{-p/2} b_\alpha)\|_p^2 = \sum_{\alpha\in\ell_A} a_\alpha^{-(q-p)}
\end{aligned}$$
yields the requested result.

□

Corollary 2.5. *A sufficient condition for the space $\mathscr{F}_{a,b}$ to be nuclear is that*
$$\sum_{\alpha\in\ell_A} a_\alpha^{-d} < \infty$$
for some $d \in \mathbb{N}$.

Definition 2.6. *Let $(a_\alpha)_{\alpha\in\ell_A}$ be a d-regular admissible family of positive numbers. Let $(b_\alpha)_{\alpha\in\ell_A}$ such that for all $p \in \mathbb{N}$, $a_\alpha^p b_\alpha \geq 1$, for all but finitely many α. The space $\mathscr{F}'_{a,b}$ is called a d-regular admissible space. The space is called regular admissible if it is d-regular admissible for some $d \in \mathbb{N}$.*

We denote by $\mathscr{F}'_{a,b,p}$ the dual of $\mathscr{F}_{a,b,p}$. Then

$$\mathscr{F}'_{a,b,1} \subseteq \mathscr{F}'_{a,b,2} \subseteq \cdots \subseteq \mathscr{F}'_{a,b,p} \subseteq \cdots \subseteq \mathscr{F}'_{a,b},$$

and the dual space $\mathscr{F}'_{a,b}$ is the union of the increasing sequence of the spaces $\mathscr{F}'_{a,b,p}$, i.e.,

$$\mathscr{F}'_{a,b} = \bigcup_{p\in\mathbb{N}} \mathscr{F}'_p.$$

See [11]. Since a Fréchet space is nuclear if and only if its strong dual is nuclear, $\mathscr{F}'_{a,b}$ is also nuclear.

Proposition 2.7. *$\mathscr{F}'_{a,b}$ can be viewed as*

$$\left\{ (f_\alpha)_{\alpha\in\ell_A} : \sum_{\alpha\in\ell_A} |f_\alpha|^2 a_\alpha^{-p} < \infty \textit{ for some } p \in \mathbb{N} \right\}.$$

Proof. Let $f \in \mathscr{F}'_{a,b,p}$. it follows from Riesz's representation theorem that there exists $\psi = (\psi_\alpha) \in \mathscr{F}_{a,b,p}$ with $\|\psi\|_{\mathscr{F}_{a,b,p}} = \|f\|_{\mathscr{F}'_{a,b,p}}$ and such that

$$f(\cdot) = \langle \cdot, \psi \rangle_{\mathscr{F}_{a,b,p}}.$$

Thus, for every $\varphi = (\varphi_\alpha) \in \mathscr{F}_{a,b,p}$

$$f(\varphi) = \langle \psi, \varphi \rangle_{\mathscr{F}_{a,b,p}} = \sum_{\alpha\in\ell_A} \varphi_\alpha \overline{\psi_\alpha} b_\alpha^2 a_\alpha^p.$$

Setting $f_\alpha = \psi_\alpha b_\alpha a_\alpha^p$, we have

$$\sum_{\alpha\in\ell_A} |f_\alpha|^2 a_\alpha^{-p} = \sum_{\alpha\in\ell_A} |\psi_\alpha|^2 b_\alpha^2 a_\alpha^p = \|\psi\|^2_{\mathscr{F}_{a,b,p}} = \|f\|^2_{\mathscr{F}'_{a,b,p}}.$$

Moreover, for any (f_α) subjects to $\sum_{\alpha\in\ell_A} |f_\alpha|^2 a_\alpha^{-p} < \infty$

$$(f_\alpha) \mapsto \left((\varphi_\alpha) \mapsto \sum_{\alpha\in\ell_A} \varphi_\alpha \overline{f_\alpha} b_\alpha \right)$$

maps (f_α) to a continuous linear functional over $\mathscr{F}_{a,b,p}$, and any composition of this mapping with $f \mapsto (f_\alpha)$, which was described before, yields the appropriate identity. Hence,

$$\mathscr{F}'_{a,b,p} \cong \left\{ (f_\alpha)_{\alpha\in\ell_A} : \sum_{\alpha\in\ell_A} |f_\alpha|^2 a_\alpha^{-p} < \infty \right\}.$$

Thus, $\mathscr{F}'_{a,b}$ can be viewed as

$$\left\{ (f_\alpha)_{\alpha\in\ell_A} : \sum_{\alpha\in\ell_A} |f_\alpha|^2 a_\alpha^{-p} < \infty \text{ for some } p \in \mathbb{N} \right\}.$$

□

We note that the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}_b}$ coincides with the antilinear duality $\langle\cdot,\cdot\rangle_{\mathscr{F}'_{a,b},\mathscr{F}_{a,b}}$, whenever it makes sense. Considering the inclusion of dual spaces $\mathcal{H}'_b$ in $\mathscr{F}'_{a,b}$, using Riesz theorem we have that,

$$\mathscr{F}_{a,b} \subseteq \mathcal{H}_b \subseteq \mathscr{F}'_{a,b}.$$

It is clear that the second inclusion is also continuous, since

$$\|(\xi_\alpha)_{\alpha\in\ell_A}\|^2_{\mathscr{F}'_{a,b,p}} = \sum_{\alpha\in\ell_A} |\xi_\alpha|^2 a_\alpha^{-p} \leq \sum_{\alpha\in\ell_A} |\xi_\alpha|^2 b_\alpha = \|(\xi_\alpha)_{\alpha\in\ell_A}\|^2_{\mathcal{H}_b}.$$

Proposition 2.8. *$(\mathscr{F}_{a,b}, \mathcal{H}_b, \mathscr{F}'_{a,b})$ is a Gelfand triple.*

3. Våge spaces - a new family of topological rings

Definition 3.1 below reduces to the convolution and the product introduced by Hida and Ikeda, see [14], respectively in the special cases mentioned after Definition 2.1. Following Hida and Ikeda, we will also call *Wick product* the product (3.1).

Definition 3.1. *Let $A \subset \mathbb{N}$ and define ℓ_A by (2.1). The Wick product $f \diamond g$ of two families of complex numbers $f = (f_\alpha)_{\alpha\in\ell_A}$ and $g = (g_\beta)_{\beta\in\ell_A}$ is defined by*

$$f \diamond g = \left(\sum_{\beta\leq\alpha} f_\beta g_{\alpha-\beta}\right)_{\alpha\in\ell_A}. \tag{3.1}$$

The proof of the following proposition is straightforward and will be omitted.

Proposition 3.2. *Let f, g and h be in $\mathbb{C}^{\ell_A}$. Then, it holds that:*
(a) $f \diamond g = g \diamond f$.
(b) $(f \diamond g) \diamond h = f \diamond (g \diamond h)$.
(c) $f \diamond (g + h) = f \diamond g + f \diamond h$.

Recall that we introduced regular admissible spaces in Definition 2.6.

Definition 3.3. *A regular admissible space $\mathscr{F}'_{a,b} = \bigcup_{p=1}^{\infty} \mathscr{F}'_{a,b,p}$ is called a Våge space if there is $e \in \mathbb{N}$ such that for every $p \in \mathbb{N}$ and for every $p \geq q + e$*

$$\|f \diamond g\|_p \leq A(p-q)\|f\|_q\|g\|_p$$

for all $f \in \mathscr{F}'_{a,b,q}$ and $g \in \mathscr{F}'_{a,b,p}$, whereas $A(p-q)$ is a finite positive number.

When $\mathscr{F}_{a,b}$ is a Våge space, we call the minimal e with this property the *index* of the space.

Theorem 3.4. *A d-regular admissible space $\mathscr{F}'_{a,b}$ is a Våge space if and only if*

$$a_\alpha a_\beta \leq a_{\alpha+\beta}, \quad \forall \alpha, \beta \in \ell_A,$$

i.e., $(a_\alpha)_{\alpha\in\ell_A}$ is superexponential. Its index is then less or equal to d.

Proof. We follow the argument in [18, p. 118]. First we assume that for all $\alpha, \beta \in \ell_A$, $a_\alpha a_\beta \leq a_{\alpha+\beta}$. Since $\mathscr{F}'_{a,b}$ is a regular admissible space, $a_0 = 1$, $a_{e_n} > 1$ for all $n \in A$, and $\sum_{n\in A} \frac{1}{a_{e_n}^d - 1} < \infty$ for some $d \in \mathbb{N}$. Therefore by Proposition 2.2 we obtain $\sum_{\alpha\in\ell_A} a_\alpha^{-(p-q)} \leq \sum_{\alpha\in\ell_A} a_\alpha^{-d} < \infty$. We denote

$$A(p-q) = \left(\sum_{\alpha\in\ell_A} a_\alpha^{-(p-q)}\right)^{\frac{1}{2}}.$$

Now, supposed that $f = (f_\alpha)_{\alpha\in\ell_A} \in \mathscr{F}'_{a,b,q}$ and $g = (g_\beta)_{\beta\in\ell_A} \in \mathscr{F}_{a,b,p}$, for some $p \geq q+d$. Then,

$$\begin{aligned}
\|f \diamond g\|_p^2 &= \sum_{\gamma\in\ell_A} \left|\sum_{\alpha\leq\gamma} f_\alpha g_{\gamma-\alpha} a_\gamma^{-p/2}\right|^2 \\
&\leq \sum_{\gamma\in\ell_A} \left(\sum_{\alpha\leq\gamma} |f_\alpha| a_\alpha^{-p/2} |g_{\gamma-\alpha}| a_{\gamma-\alpha}^{-p/2}\right)^2 \\
&\leq \sum_{\gamma\in\ell_A} \left(\sum_{\alpha,\alpha'\leq\gamma} |f_\alpha| a_\alpha^{-p/2} |f_{\alpha'}| a_{\alpha'}^{-p/2} |g_{\gamma-\alpha}| a_{\gamma-\alpha}^{-p/2} |g_{\gamma-\alpha'}| a_{\gamma-\alpha'}^{-p/2}\right) \\
&\leq \sum_{\alpha,\alpha'\in\ell_A} \left(|f_\alpha| a_\alpha^{-p/2} |f_{\alpha'}| a_{\alpha'}^{-p/2} \sum_{\gamma\geq\alpha,\alpha'} |g_{\gamma-\alpha}| a_{\gamma-\alpha}^{-p/2} |g_{\gamma-\alpha'}| a_{\gamma-\alpha'}^{-p/2}\right) \\
&\leq \left(\sum_{\beta\in\ell_A} |f_\beta| a_\beta^{-p/2}\right)^2 \left(\sum_{\beta\in\ell_A} |g_\beta|^2 a_\beta^{-p}\right)^{\frac{1}{2}} \left(\sum_{\beta\in\ell_A} |g_\beta|^2 a_\beta^{-p}\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{\beta\in\ell_A} a_\beta^{-(p-q)}\right) \left(\sum_{\beta\in\ell_A} |f_\beta|^2 a_\beta^{-q}\right) \left(\sum_{\beta\in\ell_A} |g_\beta|^2 a_\beta^{-p}\right) \\
&= (A(p-q))^2 \, \|f\|_q^2 \|g\|_p^2.
\end{aligned}$$

Thus, $\mathscr{F}_{a,b}$ is a Våge space of index which is less than or equal to d. On the other direction, assuming that $\mathscr{F}'_{a,b}$ is a Våge space of index e.

Let q be a natural number and $p = q + e$. Then,

$$\|\delta_{\alpha+\beta}\|_p = \|\delta_\alpha \diamond \delta_\beta\|_p \le A(p-q)\|\delta_\alpha\|_q\|\delta_\beta\|_p.$$

Therefore, $a_{\alpha+\beta}^{-p} \le A(e)a_\alpha^{-p}a_\beta^{-q}$. Then, $A(e)^{-1/p}a_\alpha a_\beta^{q/p} \le a_{\alpha+\beta}$. Thus, we obtain $a_\alpha a_\beta \le a_{\alpha+\beta}$ as p goes to infinity. $\square$

Corollary 3.5. *A Våge space is nuclear.*

Proof. If $\mathscr{F}'_{a,b}$ is a Våge space then $\sum_{\alpha\in\ell_A} a_\alpha^{-d} < \infty$. Applying Corollary 2.5, $\mathscr{F}_{a,b}$ is nuclear. However, since it is a Fréchet space, $\mathscr{F}'_{a,b}$ is also nuclear. $\square$

We now show that the Wick product is continuous in the strong topology.

Proposition 3.6. *Let (f_λ) be a net in $\mathscr{F}'_{a,b}$. Then $f_\lambda \to f$ in the strong topology if and only if there exists $p \in \mathbb{N}$ such that $f_\lambda, f \in \mathscr{F}'_{a,b,p}$ and $f_\lambda \to f$ in the strong topology of $\mathscr{F}'_{a,b,p}$.*

Proof. Suppose $f_\lambda \to f$ in the strong topology of $\mathscr{F}'_{a,b}$. In particular, $\{f_\lambda\}_{\lambda\in\Lambda} \cup \{f\}$ is strongly bounded. Therefore, there exists $p \in \mathbb{N}$ such that $\{f_\lambda\}_{\lambda\in\Lambda} \cup \{f\} \subseteq \mathscr{F}'_{a,b,p}$ (see [11, §5.3 p. 45]). Let B be a bounded set in $\mathscr{F}_{a,b,p}$, then $B \cap \mathscr{F}_{a,b}$ is a dense subset of B. Therefore, $\sup_{\varphi\in B}|f_\lambda(\varphi) - f(\varphi)| = \sup_{\varphi\in B\cap\mathscr{F}_{a,b}}|f_\lambda(\varphi) - f(\varphi)| \to 0$. Thus, $f_\lambda \to f$ in the strong topology of $\mathscr{F}'_{a,b,p}$. The opposite direction is clear. $\square$

Theorem 3.7. *Let $\mathscr{F}'_{a,b}$ be a Våge space. Then the Wick product is a continuous function $\mathscr{F}'_{a,b} \times \mathscr{F}'_{a,b} \to \mathscr{F}'_{a,b}$ in the strong topology.*

Proof. Assuming $((f_\lambda, g_\lambda))_{\lambda\in\Lambda}$ is a net which converges to (f, g) in the strong topology of $\mathscr{F}'_{a,b} \times \mathscr{F}'_{a,b}$, then in particular, $f_\lambda \to f$ and $g_\lambda \to g$ in the strong topology of $\mathscr{F}'_{a,b}$. According to Proposition 3.6, there exist $p, q \in \mathbb{N}$ such that $f_\lambda, f \in \mathscr{F}'_{a,b,q}$ and $g_\lambda, g \in \mathscr{F}'_{a,b,p}$ where $f_\lambda \to f$ in the strong topology of $\mathscr{F}'_{a,b,q}$ and $g_\lambda \to g$ in the strong topology of $\mathscr{F}'_{a,b,p}$. We may assume that $p \ge q + d$. Since $\mathscr{F}'_{a,b}$ is a Våge space, $f \diamond g_\lambda, f \diamond g \in \mathscr{F}'_{a,b,p}$, and $\diamond : \mathscr{F}'_{a,b,q} \times \mathscr{F}'_{a,b,p} \to \mathscr{F}'_{a,b,p}$ is continuous. Since $(f_\lambda, g_\lambda) \to (f, g)$ in the strong topology of $\mathscr{F}'_{a,b,q} \times \mathscr{F}'_{a,b,p}$, $f_\lambda \diamond g_\lambda \to f \diamond g$ in the strong topology of $\mathscr{F}'_{a,b,p}$. Again, using Proposition 3.6, we have that $f_\lambda \diamond g_\lambda \to f \diamond g$ in the strong topology of $\mathscr{F}'_{a,b}$. Thus the Wick product is strongly continuous. $\square$

In conclusion, $(\mathscr{F}'_{a,b}, +, \diamond)$ is a commutative topological ring. We will denote it by $\mathcal{R}$.

We do not know if the Wick product is continuous in the weak topology. We end this section with the weak topology analogue of Proposition 3.6.

Proposition 3.8. *Let (f_λ) be a net in $\mathscr{F}'_{a,b}$. Then $f_\lambda \to f$ in the weak topology if and only if there exists $p \in \mathbb{N}$ such that $f_\lambda, f \in \mathscr{F}'_{a,b,p}$ and $f_\lambda \to f$ in the weak topology of $\mathscr{F}'_{a,b,p}$.*

Proof. Suppose $f_\lambda \to f$ in the weak topology .In particular, $\{f_\lambda\} \cup \{f\}$ is weakly bounded, and thus strongly bounded (see [11, p. 48]). Therefore, there exists $p \in \mathbb{N}$ such that $\{f_\lambda\} \cup \{f\} \subseteq \mathscr{F}'_{a,b,p}$. Moreover, $f_\lambda \to f$ pointwise on a dense subset of $\mathscr{F}_{a,b,p}$, that is $\mathscr{F}_{a,b}$. Let $\epsilon > 0$ and $\varphi \in \mathscr{F}_{a,b,p}$. We may choose $\psi \in \mathscr{F}_{a,b}$ such that $\|\varphi - \psi\|_p < \frac{\epsilon}{2(\|f\| + \sup_\lambda \|f_\lambda\|)}$, and $\lambda_0 \in \Lambda$ such that for all $\lambda \geq \lambda_0$, $|f_\lambda(\psi) - f(\psi)| < \frac{\epsilon}{2}$. Therefore,

$$\begin{aligned} |f_\lambda(\varphi) - f(\varphi)| &\leq |f_\lambda(\varphi) - f_\lambda(\psi)| + |f_\lambda(\psi) - f(\psi)| + |f(\psi) - f(\varphi)| \\ &\leq (\|f_\lambda\| + \|f\|)\|\varphi - \psi\| + |f_\lambda(\psi) - f(\psi)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus $f_\lambda \to f$ in the weak topology of $\mathscr{F}'_{a,b,p}$. The opposite direction is clear. ∎

4. Invertible elements and power series

Definition 4.1. *Let $f = (f_\alpha)_{\alpha \in \ell_A} \in \mathcal{R}$. Then, $f_0 \in \mathbb{C}$ is called the generalized expectation of f and is denoted by $E[f]$.*

From this definition we have that

$$E[f \diamond g] = E[f]E[g], \quad \forall f, g \in \mathcal{R}.$$

We note that $E : \mathcal{R} \to \mathbb{C}$ is a homomorphism which maps $1_{\mathcal{R}}$ to $1_{\mathbb{C}}$. In the sequel, we will see it is the only homomorphism with this property (see Proposition 4.7).

Proposition 4.2. *Let M be a positive number. Then, for any $f \in \mathcal{R}$ such that $E[f] = 0$ there is $q \in \mathbb{N}$ such that $\|f\|_q < M$.*

Proof. Let $f = (f_\alpha) \in \mathscr{F}'_{a,b,p}$ with $f_0 = 0$. Since $a_{e_n} > 1$ for $n = 1, 2, \ldots$, we have

$$a_\alpha = \prod_{n \in A} a_{e_n}^{\alpha_n} > 1 \qquad \text{for all } \alpha \neq (0, 0, \ldots).$$

Therefore, for all $\alpha \in \ell_A$ $\lim_{q \to \infty} |f_\alpha|^2 a_\alpha^{-q} = 0$ (recall $f_0 = 0$) and for all $q > p$, $|f_\alpha|^2 a_\alpha^{-q} \leq |f_\alpha|^2 a_\alpha^{-p}$, whereas $\sum_{\alpha \in \ell_A} |f_\alpha|^2 a_\alpha^{-p} = \|f\|_p^2 < \infty$.

Thus, the dominated convergence theorem implies

$$\lim_{q\to\infty} \|f\|_q^2 = \lim_{q\to\infty} \sum_{\alpha\in\ell_A} |f_\alpha|^2 a_\alpha^{-q} = \sum_{\alpha\in\ell_A} \lim_{q\to\infty} |f_\alpha|^2 a_\alpha^{-q} = 0.$$

□

Definition 4.3. *The n-th Wick power $f^{\diamond n}$ of f is defined inductively as follows:*

$$f^{\diamond n} = \begin{cases} 1 & \text{if } n = 0 \\ f \diamond f^{\diamond(n-1)} & \text{if } n > 0 \end{cases}$$

Proposition 4.4. *Let $\mathcal{R}$ be a Våge space of index d and let f be in $\mathscr{F}'_{a,b,p}$. Then $\forall n \in \mathbb{N}$, $f^{\diamond n} \in \mathscr{F}'_{a,b,p+d}$. Moreover, $\|f^{\diamond n}\|_{p+d} \leq A(d)^n \|f\|_p^n$.*

Proof. Obviously, $f^{\diamond 0} = 1 \in \mathscr{F}'_{a,b,p+d}$, and $\|f^{\diamond 0}\|_{p+d} = A(d)^0 \|f\|_p^0$.
By induction, if we assume $f^{\diamond n} \in \mathcal{R}$, we get $f^{\diamond(n+1)} = f \diamond f^{\diamond n} \in \mathcal{R}$, and

$$\begin{aligned} \|f^{\diamond(n+1)}\|_{p+d} &= \|f \diamond f^{\diamond n}\|_{p+d} \\ &\leq A(d)\|f\|_p \|f^{\diamond n}\|_{p+d} \\ &\leq A(d)^n \|f\|_p^{n+1} < \infty \end{aligned}$$

□

More generally, given a polynomial $p(z) = \sum_{n=0}^N p_n z^n$ ($p_n \in \mathbb{C}$), we define its Wick version

$$p^\diamond : \mathcal{R} \to \mathcal{R}$$

by

$$p^\diamond(f) = \sum_{n=0}^N p_n f^{\diamond n}$$

By Proposition 4.4, we have that $p^\diamond(f) \in \mathcal{R}$ for $f \in \mathcal{R}$. The following proposition considers the case of power series.

Proposition 4.5. *Let $\phi(z) = \sum_{n\in\mathbb{N}} \phi_n z^n$ be a power series (with complex coefficients) which converges absolutely in the open disk with radius R. Then for any $f \in \mathcal{R}$ such that $|E[f]| < \frac{R}{A(d)}$, $\phi^\diamond(f) = \sum_{n\in\mathbb{N}} \phi_n f^{\diamond n} \in \mathcal{R}$.*

Proof. Applying Proposition 4.2, there exists q such that $\|f - E(f)\|_q < \frac{R}{A(d)} - |E[f]|$. Therefore,

$$\|f\|_q \leq \|f - E(f)1_{\mathcal{R}}\|_q + |E(f)| < \frac{R}{A(d)}.$$

Then by Proposition 4.4, for all $p \geq q + d$,

$$\begin{aligned}\sum_{n\in\mathbb{N}} |\phi_n| \|f^{\diamond n}\|_p &\leq \sum_{n\in\mathbb{N}} |\phi_n| A(d)^n \|f\|_q^n \\ &= \sum_{n\in\mathbb{N}} |\phi_n| (A(d)\|f\|_q)^n \\ &< \infty.\end{aligned}$$

Since $\mathscr{F}_{a,b,q}$ is a Hilbert space, $\phi^\diamond(f) = \sum_{n\in\mathbb{N}} \phi_n f^{\diamond n} \in \mathscr{F}_{a,b,q}$. Thus, $\phi^\diamond(f) \in \mathcal{R}$. □

Proposition 4.6. *The element $f \in \mathcal{R}$ is invertible if and only if $E[f]$ is invertible.*

Proof. If $E[f] \neq 0$, we can assume that $E[f] = 1$. By the Proposition 4.5 we have that $\sum_{n\in\mathbb{N}}(1-f)^{\diamond n} \in \mathcal{R}$. Furthermore,

$$f \diamond \left(\sum_{n\in\mathbb{N}} (1-f)^{\diamond n} \right) = 1.$$

Conversely, assume f invertible. Then there exists $f^{-1} \in \mathcal{R}$ such that $f \diamond f^{-1} = 1$. Hence, $E[f]E[f^{-1}] = E[f \diamond f^{-1}] = 1$. □

Proposition 4.7. *Let $\mathcal{R}$ be a Våge space. Then the following properties hold:*

(a) $GL(\mathcal{R})$ is open.
(b) The spectrum of $f \in \mathcal{R}$, $\sigma(f) = \{\lambda \in \mathbb{C} : f - \lambda 1_{\mathcal{R}}$ is not invertible $\}$ is the singleton $\{E[f]\}$.
(c) E is unique as a homomorphism $\mathcal{R} \to \mathbb{C}$ mapping $1_{\mathcal{R}}$ to $1_{\mathbb{C}}$.

Proof.

(a) By Proposition 4.6, we have that $\{f \in \mathcal{R} : E[f] \neq 0\}$ is the set of all invertible elements in $\mathcal{R}$. In other words, $GL(\mathcal{R}) = E^{-1}(GL(\mathbb{C}))$. In particular, since E is continuous, $GL(\mathcal{R})$ is open.
(b) Clearly, $f - \lambda 1_{\mathcal{R}}$ does not have an inverse if and only if $\lambda = E(f)$.
(c) Let $\varphi : \mathcal{R} \to \mathbb{C}$ be a homomorphism mapping $1_{\mathcal{R}}$ to $1_{\mathbb{C}}$ and let $f \in \mathcal{R}$. Since $\varphi\left(f - \varphi(f)1_{\mathcal{R}}\right) = 0$, $\varphi(f) \in \sigma(f)$, that is $\varphi(f) = E[f]$.

□

Using Proposition 4.6 we can define rational functions with coefficients in $\mathcal{R}$ as in [4] and [2, Section 3].

Definition 4.8. *A rational function with values in $\mathcal{R}^{n\times m}$ is an expression of the form*

$$R^\diamond(f) = p^\diamond(f)(q^\diamond(f))^{-1} \tag{4.1}$$

where $p \in (\mathcal{R}^\diamond[f])^{n\times m}$, *and* $0 \neq q \in \mathcal{R}^\diamond[f]$.

In Section 8 we discuss some problems in the theory of linear systems using rational functions with coefficients in $\mathcal{R}$.

To conclude this section, we remark that one may consider classical interpolation problems (of the kind developed in [6]) in the present setting. For instance one can consider the following Nevanlinna-Pick interpolation problem:

Problem 4.9. *Given* $N \in \mathbb{N}$ *and* N *pairs of points* $(a_1, b_1), \ldots, (a_N, b_N)$ *in* $\mathcal{R}^2$, *find all power series* ϕ *such that*

$$\phi^\diamond(a_j) = b_j, \quad j = 1, \ldots, N,$$

and such that, moreover, the function $z \mapsto E(\phi^\diamond(z))$ *is analytic and contractive in the open unit disk.*

This problem, as well as more general interpolation problems, have been studied in [2] when $\mathcal{R}$ is the Kondratiev space $\mathcal{S}_{-1}$. Two important tools used there, and which are still valid in the setting of Våge spaces are the permanence of algebraic identities (see [5, p. 456]) and the definition and properties of analytic functions with values in the dual of a countably normed Hilbert space.

5. Tensor product of Våge spaces

When one considers the tensor product $E \otimes F$ of two locally convex Hausdorff spaces E, F, some "natural" topologies may be considered. Two such topologies are the π-topology and the ϵ-topology (see [12], [27, Chapter 43, p. 434]). These topologized tensor products of E and F are denoted respectively by $E \otimes_\pi F$ and $E \otimes_\epsilon F$. The completions of the tensor product of E and F with respect to the π-topology and the ϵ-topology, will then be denoted by $E\hat{\otimes}_\pi F$ and $E\hat{\otimes}_\epsilon F$ respectively. However, when it comes to nuclear spaces, things are getting much easier.

Theorem 5.1. *Let* E *be a locally convex Hausdorff space. Then,* E *is nuclear if and only if for every locally convex Hausdorff space* F, $E\hat{\otimes}_\pi F = E\hat{\otimes}_\epsilon F$.

A proof can be found in [27, Theorem 50.1 p. 511]. Thanks to this last theorem, we simply denote

$$E\hat{\otimes} F \stackrel{\text{def.}}{=} E\hat{\otimes}_\pi F = E\hat{\otimes}_\epsilon F$$

when one of the spaces E or F is nuclear. We also denote the usual tensor product of two Hilbert spaces E and F by $E \otimes F$. We recall the following result of Grothendieck on tensor products, see [12]:

Proposition 5.2. *Let E be a complete locally convex space of functions defined on a set T, such that its topology is finer than the pointwise convergence topology, and assume E to be nuclear. Then for every complete locally convex space F, the tensor product $E \hat{\otimes} F$ can be interpreted as the space of all functions $f : T \to F$ such that for all $y' \in F'$, $t \mapsto \langle y', f(t) \rangle_{F',F}$ is a function of E.*

Let $(a_\alpha)_{\alpha \in \ell_A}$, $(b_\alpha)_{\alpha \in \ell_A}$, $(c_\alpha)_{\alpha \in \ell_A}$ and $(d_\alpha)_{\alpha \in \ell_A}$ be four families of positive numbers, such that the associated countably Hilbert spaces $\mathscr{F}_{a,b}$ and $\mathscr{G}_{c,d}$ are nuclear. Recall the notation

$$\mathscr{F}_{a,b,p} = \left\{ (\varphi_\alpha)_{\alpha \in \ell_A} : \sum_{\alpha \in \ell_A} |\varphi_\alpha|^2 b_\alpha^2 a_\alpha^p < \infty \right\}$$

and

$$\mathscr{G}_{c,d,q} = \left\{ (\psi_\beta)_{\beta \in \ell_B} : \sum_{\beta \in \ell_B} |\psi_\beta|^2 d_\beta^2 c_\beta^q < \infty \right\}$$

and that $\mathscr{F}_{a,b} = \bigcap_{p \in \mathbb{N}} \mathscr{F}_{a,b,p}$ and $\mathscr{G}_{c,d} = \bigcap_{q \in \mathbb{N}} \mathscr{G}_{c,d,q}$. Since $\mathscr{F}_{a,b}$ is a complete locally convex space of functions defined on the free commutative monoid ℓ_A, and since its topology is finer than the pointwise topology (that is since $|\varphi_{\alpha\lambda} - \varphi_\alpha|^2 b_\alpha^2 a_\alpha^p \leq \sum_{\alpha \in \ell_A} |\varphi_{\alpha\lambda} - \varphi_\alpha|^2 b_\alpha^2 a_\alpha^p$), $\mathscr{F}_{a,b} \hat{\otimes} \mathscr{G}_{c,d}$ can be interpreted as the space of all elements of the form $\psi_{\beta,\alpha}$ such that for all $f = (f_\beta)_{\beta \in \ell_B} \in \mathscr{G}'_{c,d}$, $(\langle f, \psi_{\beta,\alpha} \rangle_{\mathscr{G}'_{c,d}, \mathscr{G}_{c,d}})_{\alpha \in \ell_A} \in \mathscr{F}_{a,b}$.

Theorem 5.3. *It holds that*

$$\mathscr{F}_{a,b} \hat{\otimes} \mathscr{G}_{c,d} = \bigcap_{p,q} \mathscr{F}_{a,b,p} \otimes \mathscr{G}_{c,d,q}.$$

Proof. Let $(\psi_{\beta,\alpha}) \in \mathscr{F}_{a,b} \hat{\otimes} \mathscr{G}_{c,d}$ be a non-negative real sequence. Then for all $f = \{f_\beta\}_{\beta \in \ell_B} \in \mathscr{G}'_{c,d}$, $(\langle f, \psi_{\beta,\alpha} \rangle_{\mathscr{G}'_{c,d}, \mathscr{G}_{c,d}})_{\alpha \in \ell_A} \in \mathscr{F}_{a,b}$. In particular, we may choose $f_\beta = c_\beta^{\frac{q}{2}}$ for any $q \in \mathbb{N}$. Therefore,

$$\begin{aligned}\sum_{\alpha \in \ell_A} \sum_{\beta \in \ell_B} |\psi_{\beta,\alpha}|^2 d_\beta^2 c_\beta^q b_\alpha^2 a_\alpha^p &\leq \sum_{\alpha \in \ell_A} \left| \sum_{\beta \in \ell_B} c_\beta^{\frac{q}{2}} \psi_{\beta,\alpha} d_\beta \right|^2 b_\alpha^2 a_\alpha^p \\ &= \sum_{\alpha \in \ell_A} \left| \langle (c_\beta^{\frac{q}{2}}), (\psi_{\beta,\alpha}) \rangle_{\mathscr{G}'_{c,d}, \mathscr{G}_{c,d}} \right|^2 b_\alpha^2 a_\alpha^p < \infty\end{aligned}$$

Thus, $(\psi_{\beta,\alpha}) \in \bigcap_{p,q} \mathscr{F}_{a,b,p} \otimes \mathscr{G}_{c,d,q}$. In case $(\psi_{\beta,\alpha}) \in \mathscr{F}_{a,b} \hat{\otimes} \mathscr{G}_{c,d}$ is an arbitrary sequence, then applying the last inequality to positive real part,

negative real part, positive imaginary part and negative imaginary part yields the requested result.

To prove the opposite direction, take $(\psi_{\beta,\alpha}) \in \bigcap_{p,q} \mathscr{F}_{a,b,p} \otimes \mathscr{G}_{c,d,q}$. Then for all $f = (f_\beta)_{\beta \in \ell_B} \in \mathscr{G}'_{c,d}$ there exists q such that $f \in \mathscr{G}'_{c,d,q}$. Thus,

$$\begin{aligned}
\left| \langle f, (\psi_{\beta,\alpha}) \rangle_{\mathscr{G}'_{c,d}, \mathscr{G}_{c,d}} \right|^2 &= \left| \sum_{\beta \in \ell_B} \overline{f_\beta} \psi_{\beta,\alpha} d_\beta \right|^2 \\
&= \left| \sum_{\beta \in \ell_B} \overline{f_\beta} c_\beta^{-\frac{q}{2}} \psi_{\beta,\alpha} d_\beta c_\beta^{\frac{q}{2}} \right|^2 \\
&\leq \left(\sum_{\beta \in \ell_B} |f_\beta|^2 c_\beta^{-q} \right) \left(\sum_{\beta \in \ell_B} |\psi_{\beta,\alpha}|^2 d_\beta^2 c_\beta^q \right) \\
&= \|f\|^2_{\mathscr{G}'_{c,d,q}} \sum_{\beta \in \ell_B} |\psi_{\beta,\alpha}|^2 d_\beta^2 c_\beta^q
\end{aligned}$$

Hence for all $p \in \mathbb{N}$,

$$\sum_{\alpha \in \ell_A} \left| \langle f, (\psi_{\beta,\alpha}) \rangle_{\mathscr{G}'_{c,d}, \mathscr{G}_{c,d}} \right|^2 b_\alpha^2 a_\alpha^p \leq \|f\|^2_{\mathscr{G}'_{c,d,q}} \sum_{\alpha \in \ell_A} \sum_{\beta \in \ell_B} |\psi_{\beta,\alpha}|^2 d_\beta^2 c_\beta^q b_\alpha^2 a_\alpha^p < \infty$$

and so $(\psi_{\beta,\alpha}) \in \mathscr{F}_{a,b} \hat{\otimes} \mathscr{G}_{c,d}$. □

Proposition 5.4. *It holds that*

$$\bigcap_{p,q} \mathscr{F}_{a,b,p} \otimes \mathscr{G}_{c,d,q} = \bigcap_{p} \mathscr{F}_{a,b,p} \otimes \mathscr{G}_{c,d,p}$$

Proof. One direction is clear. The other direction follows from the inclusion

$$\mathscr{F}_{a,b,\max\{p,q\}} \otimes \mathscr{G}_{c,d,\max\{p,q\}} \subseteq \mathscr{F}_{a,b,p} \otimes \mathscr{G}_{c,d,q}.$$

□

Applying Theorem 5.3 and Proposition 5.4 ,we obtain:
(5.1)

$$\mathscr{F}_{a,b} \hat{\otimes} \mathscr{G}_{c,d} = \bigcap_{p \in \mathbb{N}} \left\{ (\psi_{\alpha,\beta}) : \sum_{\alpha \in \ell_A} \sum_{\beta \in \ell_B} |\psi_{\beta,\alpha}|^2 (d_\beta b_\alpha)^2 (c_\beta a_\alpha)^p < \infty \right\}.$$

Now, we can concatenate the indices in an obvious way. We define $C = 2B \cup (2A-1)$ (disjoint union), $P_B : C \to B$, $P_A : C \to A$ the appropriate projections, $g_\gamma = c_{P_B(\gamma)} a_{P_A(\gamma)}$, $h_\gamma = d_{P_B(\gamma)} b_{P_A(\gamma)}$, and

$\psi_\gamma = \psi_{P_B(\gamma),P_A(\gamma)}$. Therefore, we may write

$$\mathscr{F}_{a,b}\hat{\otimes}\mathscr{G}_{c,d} = \bigcap_{p\in\mathbb{N}} \left\{ (\psi_\gamma)_{\gamma\in\ell_C} : \sum_{\gamma\in\ell_C} |\psi_\gamma|^2 h_\gamma^2 g_\gamma^p < \infty \right\},$$

and

$$(\mathscr{F}_{a,b}\hat{\otimes}\mathscr{G}_{c,d})' = \bigcup_{p\in\mathbb{N}} \left\{ (\psi_\gamma)_{\gamma\in\ell_C} : \sum_{\gamma\in\ell_C} |\psi_\gamma|^2 g_\gamma^{-p} < \infty \right\}.$$

Proposition 5.5. *Let $(a_\alpha)_{\alpha\in\ell_A}$ and $(c_\beta)_{\beta\in\ell_B}$ be two admissible families of positive numbers. Then:*
(a) $(g_\gamma)_{\gamma\in\ell_C}$ (whereas $C = 2B \cup (2A-1)$ and $g_\gamma = c_{P_B(\gamma)}a_{P_A(\gamma)}$) is admissible.
(b) If $(a_\alpha)_{\alpha\in\ell_A}$ and $(c_\beta)_{\beta\in\ell_B}$ are both d-regular, then $(g_\gamma)_{\gamma\in\ell_C}$ is also d-regular, and if $(a_\alpha)_{\alpha\in\ell_A}$ and $(c_\beta)_{\beta\in\ell_B}$ are both superexponential, then $(g_\gamma)_{\gamma\in\ell_C}$ is also superexponential.
(c) If $(b_\alpha)_{\alpha\in\ell_A}$ and $(d_\beta)_{\beta\in\ell_B}$ are two families of positive numbers such that for all $p\in\mathbb{N}$, $a_\alpha^p b_\alpha \geq 1$ for all but finitely many α and $c_\beta^p d_\beta \geq 1$ for all but finitely many β, then for all $p \in \mathbb{N}$, $g_\gamma^p h_\gamma \geq 1$ for all but finitely many γ ($h_\gamma = d_{P_B(\gamma)}b_{P_A(\gamma)}$).

Proof. $(g_\alpha)_{\alpha\in\ell_C}$ is admissible, since $g_0 = c_0a_0 = 1$ and $g_{e_n} > 1$ for all $n \in C$. Moreover,

$$\sum_{n\in C} \frac{1}{g_{e_n}^d - 1} = \sum_{n\in A} \frac{1}{a_{e_n}^d - 1} + \sum_{n\in B} \frac{1}{c_{e_n}^d - 1},$$

and hence d-regularity of both $(a_\alpha)_{\alpha\in\ell_A}$ and $(c_\beta)_{\beta\in\ell_B}$ yields d-regularity of $(g_\alpha)_{\alpha\in\ell_C}$. The rest of the proof is clear. □

Finally, we give the following theorem.

Theorem 5.6. *Let E, F be two Fréchet spaces. If E is nuclear, we have the canonical isomorphism*

$$(E\hat{\otimes}F)' = E'\hat{\otimes}F'$$

A proof, in case both E and F are nuclear, is given in [27, (50.19), p. 525]. We can now state:

Theorem 5.7. *A tensor product of two Våge spaces is also a Våge space.*

Proof. Applying (5.1), Proposition 5.5 and Theorem 3.4, $(\mathscr{F}_{a,b}\hat{\otimes}\mathscr{G}_{c,d})'$ is a Våge space. Theorem 5.6 yields the requested result. □

6. An extension of the space of tempered distributions

In this section we consider the special case $\ell_A = \mathbb{N}_0$ (i.e. $A = \{1\}$), and

$$a_n = (n+1)^2, \quad b_n = 1.$$

The corresponding space $\mathscr{F}_{a,b}$ (defined by (2.7)) is identified with the Schwartz space $\mathscr{S}$ of rapidly decreasing smooth functions, and its dual is the space $\mathscr{S}'$ of tempered distributions. We will show (see Proposition 6.1 below) that $\mathscr{S}'$ is a regular admissible space, but it is not a Våge space. We will also construct a Våge space containing $\mathscr{S}'$.

We recall (see [23, Chapter IV, Section 2, p. 303], [26, p. 105]) that the *Hermite polynomials* $h_n(x)$ are defined by

$$h_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right), \quad n = 0, 1, \ldots, \tag{6.1}$$

Various notations and conventions are given for these polynomials. See the discussion on the end of page 105 of [26]. In particular, the multiplicative factor $(-1)^n$ (which does not appear in Sansone's book [23]) insures that the factor of x^n in h_n is positive. The *Hermite functions* $\xi_n(x)$ are defined by

$$\xi_n(x) = \pi^{-\frac{1}{4}}(2^n n!)^{-\frac{1}{2}} e^{\frac{1}{2}x^2} h_n(x), \quad n = 0, 1, 2, \ldots. \tag{6.2}$$

The Hermite functions $(\xi_n)_{n\in\mathbb{N}_0}$ form an orthonormal basis of $\mathbf{L}_2(\mathbb{R}, dx)$. The *Schwartz space* $\mathscr{S}$ of smooth rapidly decreasing functions on $\mathbb{R}$ is defined by

$$\mathscr{S} = \left\{ f \in C^\infty(\mathbb{R}) : \sup_{x\in\mathbb{R}} \left| x^p f^{(q)}(x) \right| < \infty \text{ for all } p, q \in \mathbb{N}_0 \right\}$$

The Hermite functions are elements in the Schwartz space, and we have (see [21, Theorem V.13 p. 143]):

$$\mathscr{S} = \left\{ f = \sum_{n\in\mathbb{N}_0} f_n \xi_n : \sum_{n\in\mathbb{N}_0} |f_n|^2 (n+1)^{2p} < \infty \text{ for all } p \in \mathbb{N} \right\}.$$

Identifying $\sum_{n\in\mathbb{N}_0} f_n \xi_n$ with $(f_n)_{n\in\mathbb{N}_0}$ allows to identify $\mathbf{L}_2(\mathbb{R}, dx)$ with $\mathcal{H}_b$ and $\mathscr{S}$ with $\mathscr{F}_{a,b}$.

Proposition 6.1. *$\mathscr{S}'$ is a regular admissible space which is nuclear, but is not a Våge space.*

Proof. First, we note that defining $a_n = (n+1)^2$ implies that $(a_n)_{n\in\mathbb{N}_0}$ is a 1-regular admissible sequence. It is indeed admissible since $a_0 = 1$ and $a_1 = 4 > 1$, and it is indeed 1-regular, since the sum in (2.2) is

over the finite set $A = \{1\}$ and in particular does converge. Moreover, defining $b_n = 1$ implies that for any $p \in \mathbb{N}$, $a_n^p b_n \geq 1$ for all $n \in \mathbb{N}_0$. Therefore, $\mathscr{S}'$ is a 1-regular admissible space. Since, $\sum_{n\in\mathbb{N}_0} ((n+1)^2)^{-1} < \infty$, $\mathscr{S}$ is nuclear, and hence $\mathscr{S}'$ is also nuclear (of course, the nuclearity of $\mathscr{S}$ and $\mathscr{S}'$ is a standard result; see for instance [27]). Since $(a_n)_{n\in\mathbb{N}_0}$ is not superexponential, that is

$$(n+1)^2(m+1)^2 \not\leq (n+m+1)^2,$$

$\mathscr{S}'$ is not a Våge space. □

We define the following subspace of $\mathscr{S}$

$$\mathscr{G} = \left\{ \sum_{n=0}^{\infty} f_n \xi_n : \sum_{n\in\mathbb{N}_0} |f_n|^2 2^{np} < \infty \text{ for all } p \in \mathbb{N} \right\}.$$

Proposition 6.2. *$\mathscr{G}'$ is a Våge space containing the Schwartz space $\mathscr{S}'$ of tempered distributions.*

Proof. Using the identification of $\sum_{n\in\mathbb{N}_0} f_n\xi_n$ with $(f_n)_{n\in\mathbb{N}_0}$, and defining $a_n = 2^n$ and $b_n = 1$, we have that $\mathscr{G}$ is the corresponding countably Hilbert space $\mathscr{F}_{a,b}$ associated to the sequences $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ (and as before, $\mathbf{L}_2(\mathbb{R}, dx)$ is the corresponding Hilbert space $\mathcal{H}_b$). Clearly, $(a_n)_{n\in\mathbb{N}_0}$ is a 1-regular admissible sequence. It is indeed admissible since $a_0 = 1$ and $a_1 = 2 > 1$, and it is indeed 1-regular, since the sum in (2.2) is over the finite set $A = \{1\}$ and in particular does converge. Moreover, for any $p \in \mathbb{N}$, $a_n^p b_n \geq 1$ for all $n \in \mathbb{N}_0$. Therefore, $\mathscr{G}'$ is a 1-regular admissible space. Since $(a_n)_{n\in\mathbb{N}_0}$ is an exponential sequence, $\mathscr{G}'$ is a Våge space and is in particular nuclear. Moreover, the natural embeddings $\mathscr{G} \subseteq \mathscr{S}$ and $\mathscr{S}' \subseteq \mathscr{G}'$ are clearly continuous. Hence $\mathscr{G}$ is a closed subspace of $\mathscr{S}$, and $\mathscr{S}'$ is a closed subspace of $\mathscr{G}'$. □

Theorem 6.3. *$\mathscr{G}$ is the space of all entire functions $f(z)$ such that*

$$\iint_{\mathbb{C}} |f(z)|^2 \, e^{\frac{1-2^{-p}}{1+2^{-p}}x^2 - \frac{1+2^{-p}}{1-2^{-p}}y^2} dxdy < \infty \quad \text{for all } p \in \mathbb{N}.$$

In the proof we make use of two results of Hille. The first result appear in [17, formula (1.3), p. 81] and [16, Theorem 2.2 p. 885]. For the second formula, see [17, formula (2.1) p. 82] and [15, pp. 439-440]. In that last paper one can also find a history of the formula.

Theorem 6.4. *(Hille, [17]) The domain of absolute convergence of the series $\sum_{n=0}^{\infty} F_n\xi_n(z)$ is the strip $S_\tau = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < \tau\}$, where*

$$\tau = -\limsup_{n\to\infty} (2n+1)^{-\frac{1}{2}} \log |F_n|.$$

Theorem 6.5. *(Hille, [15]) The series* $\sum_{n=0}^{\infty} \xi_n(u)\xi_n(v)s^n$ *converges for arbitrary complex values of* u *and* v *when* $|s| < 1$*, and*

$$\sum_{n=0}^{\infty} \xi_n(u)\xi_n(v)s^n = \pi^{-\frac{1}{2}}(1-s^2)^{-\frac{1}{2}} e^{-\frac{(1+s^2)(u^2+v^2)-4svu}{2(1-s^2)}}.$$

Furthermore, we make use of the easy following proposition. See [22, §6, p. 61]. The space $\mathcal{F}_\alpha$ bears various names, and in particular is called the Fock space.

Proposition 6.6. *For all* $0 < \alpha \leq 1$*,*

$$\mathcal{F}_\alpha = \left\{ f \text{ is entire } : \frac{\alpha}{\pi} \iint_{\mathbb{C}} |f(z)|^2 e^{-\alpha|z|^2} dxdy < \infty \right\}$$

is a Hilbert space with a reproducing kernel $K_\alpha(z,w) = e^{\alpha \overline{w} z}$*.*

Proof of Theorem 6.3. Let $p \in \mathbb{N}$. For each $f \in \mathscr{G}$, $f = \sum_{n=0}^{\infty} f_n \xi_n$ whereas $\sum_{n=0}^{\infty} |f_n|^2 2^{np} < \infty$ for all $p \in \mathbb{N}$. In particular, $\lim_{n\to\infty} |f_n|^2 2^n = 0$. Therefore, for every n large enough, $\log|f_n| < -n \log\sqrt{2}$. Thus, in the notations of Theorem 6.4,

$$\tau = -\limsup_{n\to\infty} (2n+1)^{-\frac{1}{2}} \log|f_n| \geq \liminf_{n\to\infty} (2n+1)^{-\frac{1}{2}} n \log\sqrt{2} = \infty.$$

Therefore, denoting by $\mathscr{G}_p$ the space of all functions $f = \sum_{n=0}^{\infty} f_n \xi_n$ subject to $\sum_{n=0}^{\infty} |f_n|^2 2^{np} < \infty$, it is a Hilbert space of entire functions, and in particular, $\mathscr{G} = \bigcap_{p\in\mathbb{N}} \mathscr{G}_p$ is a countably Hilbert space of entire functions.

Now, since $\left\{\xi_n 2^{-\frac{np}{2}}\right\}_{n\in\mathbb{N}_0}$ is an orthonormal basis for $\mathscr{G}_p$, and denoting $s = 2^{-p}$, the reproducing kernel of the Hilbert space $\mathscr{G}_p$ is given by

$$G(z,w) = \sum_{n=0}^{\infty} \xi_n(z)\overline{\xi_n(w)} 2^{-np} = \sum_{n=0}^{\infty} \xi_n(z)\xi_n(\overline{w}) s^n.$$

Applying Theorem 6.5, we have that

$$G(z,w) = \pi^{-\frac{1}{2}}(1-s^2)^{-\frac{1}{2}} e^{-\frac{(1+s^2)(z^2+\overline{w}^2)-4sz\overline{w}}{2(1-s^2)}}$$

Denoting $r(z) = \pi^{-\frac{1}{4}}(1-s^2)^{-\frac{1}{4}} e^{-\frac{1+s^2}{2(1-s^2)}z^2}$, considering the kernel $K_\alpha(z,w) = e^{\alpha\overline{w}z}$ with its associated Hilbert space $\mathcal{F}_\alpha$ for $\alpha = \frac{2s}{1-s^2}$ (see Proposition 6.6), we have that

$$G(z,w) = r(z) K_\alpha(z,w) r(\overline{w}).$$

Therefore, the space $\mathscr{G}_p$ is equal to the space of functions of the form $f = rg$, with $g \in \mathcal{F}_\alpha$ and norm

$$\|f\|_{\mathscr{G}_p} = \|g\|_{\mathcal{F}_\alpha}.$$

We note that for $z = x + iy$,

$$\begin{aligned} 2 \cdot \frac{1+s^2}{2(1-s^2)}\mathrm{Re}(z^2) - \frac{2s}{1-s^2}|z|^2 &= \frac{(1+s^2)(x^2-y^2)}{1-s^2} - \frac{2s(x^2+y^2)}{1-s^2} \\ &= \frac{1-s}{1+s}x^2 - \frac{1+s}{1-s}y^2. \end{aligned}$$

Thus,

$$|r(z)|^{-2} = \sqrt{\pi(1-s^2)}e^{\frac{1-s}{1+s}x^2 - \frac{1+s}{1-s}y^2},$$

and, with $K_p = \frac{2^{1-p}}{\sqrt{\pi(1-2^{-2p})}}$,

$$\mathscr{G}_p = \left\{ f \text{ is entire } : \|f\|^2_{\mathscr{G}_p} = K_p \iint_{\mathbb{C}} |f(z)|^2 \, e^{\frac{1-2^{-p}}{1+2^{-p}}x^2 - \frac{1+2^{-p}}{1-2^{-p}}y^2} dxdy < \infty \right\},$$

and in particular $\mathscr{G} = \bigcap_{p\in\mathbb{N}} \mathscr{G}_p$ is the space of all entire functions $f(z)$ subject to

$$\iint_{\mathbb{C}} |f(z)|^2 \, e^{\frac{1-2^{-p}}{1+2^{-p}}x^2 - \frac{1+2^{-p}}{1-2^{-p}}y^2} dxdy < \infty \quad \text{for all } p \in \mathbb{N}.$$

□

We note that there are functions in the Schwartz space $\mathscr{S}$ which have no analytic continuation to an entire function but only to a holomorphic function on some strip. For example, one may consider the function

$$F(x) = \sum_{n\in\mathbb{N}_0} e^{-\sqrt{2n+1}}\xi_n(x) \in \mathscr{S}$$

It is indeed in the Schwartz space since $\sum_{n\in\mathbb{N}_0} e^{-2\sqrt{2n+1}}(n+1)^{2p} < \infty$ for all $p \in \mathbb{N}$. However,

$$\tau = -\limsup_{n\to\infty}(2n+1)^{-\frac{1}{2}} \log|F_n| = \liminf_{n\to\infty}(2n+1)^{-\frac{1}{2}}(2n+1)^{\frac{1}{2}} = 1.$$

Furthermore, clearly there are functions in the Schwartz space which have no analytic continuation to any holomorphic function, e.g., functions with compact support.

Remark 6.7. *As mentioned in the introduction, connections with the theory of hyperfunctions will be considered elsewhere.*

Remark 6.8. *Having now $\mathscr{G}'$ at hand, one can consider the tensor product $\mathscr{G}' \otimes \mathcal{S}_{-1}$, where $\mathcal{S}_{-1}$ is the Kondratiev space of stochastic distributions (see Section 7). By Section 5 it is a Våge space, and can be an appropriate setting to study stochastic linear systems. This will be done in a future publication.*

Remark 6.9. *If we define $a_0 = 1$ and for $n > 0$ $a_n = 2^{2^n}$, then $(a_n)_{n\in\mathbb{N}}$ is again a 1-regular admissible sequence. Defining $b_n = 1$, the associated countably Hilbert space $\mathscr{F}_{a,b}$ is a space of entire function. Moreover, since the sequence $(a_n)_{n\in\mathbb{N}}$ is superexponential its dual is clearly a Våge space. We can define other Våge spaces in a similar manner.*

7. The Kondratiev spaces

In this section we consider the special case $\ell_A = \ell$ (i.e. $A = \mathbb{N}$), and

$$a_\alpha = (2\mathbb{N})^\alpha, \quad b_\alpha = \alpha!.$$

Then the corresponding space $\mathscr{F}_{a,b}$ (defined by (2.7)) is identified with the Kondratiev space of Gaussian test functions $\mathcal{S}_1$, and its dual is the Kondratiev space of Gaussian stochastic distributions $\mathcal{S}_{-1}$. We will show (see Proposition 6.1 below) that $\mathcal{S}_{-1}$ is a Våge space. We also consider the Kondratiev space of Poissonian stochastic distributions, and show that it is also a Våge space.

We first need to recall a few definitions pertaining to the white noise space. The function $e^{-\frac{1}{2}\|\varphi\|^2_{\mathbf{L}_2(\mathbb{R},dx)}}$ is positive definite on the Schwartz space of real-valued functions $\mathscr{S}_\mathbb{R}$, and continuous at the origin. The Bochner-Minlos theorem (see for instance [24, pp. 10-11]) insures the existence of a probability measure $d\mu$ on the Borel σ-algebra of the dual space $\mathscr{S}'_\mathbb{R}$, such that

$$e^{-\frac{1}{2}\|\varphi\|^2_{\mathbf{L}_2(\mathbb{R},dx)}} = \int_{\mathscr{S}'_\mathbb{R}} e^{i\langle f,\varphi\rangle} d\mu(f),$$

where the brackets denote the duality between $\mathscr{S}_\mathbb{R}$ and $\mathscr{S}'_\mathbb{R}$. This equality induces an isometric map

$$s \mapsto Q_s, \quad \text{where} \quad Q_s(f) = \langle f, s\rangle$$

from $\mathscr{S}_\mathbb{R} \subset \mathbf{L}_2(\mathbb{R}, dx)$ into $\mathbf{L}_2(\mathscr{S}'_\mathbb{R}, \mathcal{B}, \mu)$.

The space $\mathcal{W} = \mathbf{L}_2(\mathscr{S}'_\mathbb{R}, \mathcal{B}, \mu)$ is called the Gaussian white noise space. We recall that the *Hermite polynomial functionals* $(H_\alpha)_{\alpha\in\ell}$

$$H_\alpha(\omega) = \prod_{k=1}^{\infty} h_{\alpha_k}(Q_{\xi_{k-1}}(\omega)).$$

where we recall that h_k and ξ_k denote respectively the Hermite function and the Hermite polynomials (see (6.1) and (6.2)), form an orthogonal basis of $\mathcal{W}$. More precisely,

$$\mathcal{W} = \left\{ \sum_{\alpha\in\ell} f_\alpha H_\alpha : \sum_{\alpha\in\ell} |f_\alpha|^2 \alpha! < \infty \right\}. \tag{7.1}$$

The Kondratiev space of Gaussian test function $\mathcal{S}_1$ is defined by

$$\mathcal{S}_1 = \left\{ \sum_{\alpha\in\ell} f_\alpha H_\alpha : \sum_{\alpha\in\ell} |f_\alpha|^2 (2\mathbb{N})^{\alpha p} (\alpha!)^2 < \infty \text{ for all } p \in \mathbb{N} \right\}.$$

Therefore, defining $a_\alpha = (2\mathbb{N})^\alpha$ and $b_\alpha = \alpha!$ allows to identify $\mathcal{S}_1$ with the corresponding countably Hilbert space $\mathscr{F}_{a,b}$ (see 2.7) and $\mathcal{W}$ and the corresponding Hilbert space $\mathcal{H}_b$ (see 2.6). We obtain a result of Våge, proved in 1996, see [28] and [18, p. 118].

Proposition 7.1. *The Kondratiev space of Gaussian stochastic distributions is a Våge space.*

Proof. We note that $(a_\alpha)_{\alpha\in\ell}$ is a 2-regular admissible family of positive numbers. It is admissible since $a_0 = (2n)^0 = 1$ and $a_{e_n} = 2n > 1$, and it is 2-regular since $\sum_{n\in\mathbb{N}} \frac{1}{(2n)^2-1} < \infty$. Therefore, and since for all $p \in \mathbb{N}$, $a_\alpha^p b_\alpha \geq 1$ for all $\alpha \in \ell$, $\mathcal{S}_{-1}$ is a 2-regular admissible space. Since $(a_\alpha)_{\alpha\in\ell}$ is exponential, $\mathcal{S}_{-1}$ is a Våge space. $\square$

Hida's theory can also be applied to Poisson processes. One then considers the function

$$\exp\left[\int_{\mathbb{R}} (e^{i\varphi(x)} - 1)dx\right]$$

which is positive definite on $\mathscr{S}_{\mathbb{R}}$, and continuous at the origin. Here too, the Bochner-Minlos theorem insures the existence of a probability measure π on $\mathscr{S}'_{\mathbb{R}}$ and such that

$$\exp\left[\int_{\mathbb{R}} (e^{i\varphi(x)} - 1)dx\right] = \int_{\mathscr{S}'_{\mathbb{R}}} e^{i\langle f,\varphi\rangle} d\pi(f).$$

The Poissonian white noise space is $\mathcal{W}^\pi = \mathbf{L}_2(\mathscr{S}'_{\mathbb{R}}, \mathcal{B}, \pi)$, and admits a representation of the form (7.1), replacing the Hermite polynomial functionals $(H_\alpha)_{\alpha\in\ell}$ with the Charlier polynomial functionals $(C_\alpha)_{\alpha\in\ell}$, computed in terms of the Poisson-Charlier polynomials; see [18, p. 185];

we refer to [26, Chapter II, §2.81, p. 34-35] for the Poisson-Charlier polynomials. More precisely,

$$\mathcal{W}^{\pi}=\left\{\sum_{\alpha\in\ell} f_\alpha C_\alpha : \sum_{\alpha\in\ell}|f_\alpha|^2\alpha! < \infty\right\}.$$

The Kondratiev space of Poissonian test function $\mathcal{S}_1^{\pi}$ is defined by

$$\mathcal{S}_1^{\pi}=\left\{\sum_{\alpha\in\ell} f_\alpha C_\alpha : \sum_{\alpha\in\ell}|f_\alpha|^2(2\mathbb{N})^{\alpha p}(\alpha!)^2 < \infty \text{ for all } p\in\mathbb{N}\right\}.$$

Since the associated families of positive numbers $(a_\alpha)_{\alpha\in\ell}$ and $(b_\alpha)_{\alpha\in\ell}$ remains the same as before, we conclude the following proposition.

Proposition 7.2. *The Kondratiev space of Poissonian stochastic distributions is a Våge space.*

We refer to [19, Theorem 3.7 p. 192] for another example of a space where Våge inequality holds in the setting of white noise space analysis.

8. State space theory and Våge spaces

The results presented in [4, 2] for the case of the Kondratiev space $\mathcal{S}_{-1}$ of stochastic distributions extend to general Våge spaces. We refer to [9, 10, 20]for general background on the theory of linear systems when the coefficient space is $\mathbb{C}$ (we also refer to [1] for a survey), and to the papers [25, 13] and to the book [7] for more information on linear system on commutative rings, and in particular for the notions of controllable and observable pairs. These various notions are also reviewed in [4].

In this section, up to the concluding remarks, we omit the notation $\diamond$, i.e., we simply write fg rather than $f\diamond g$, f^n rather than $f^{\diamond n}$, etc. We begin this section with the following proposition.

Proposition 8.1. *A matrix-valued rational function $R(z)=p(z)(q(z))^{-1}$ for which $E(q(0))\neq 0$ can be written as*

$$\hat{h}(z)=D+zC(I-zA)^{-1}B \tag{8.1}$$

where A, B, C, and D are matrices of appropriate dimensions and with entries in the ring $\mathcal{R}$.

We note that one can compute the value of a rational function of the form (4.1) at every point $f\in\mathcal{R}$ such that $E(q(f))\neq 0$. Let $q(z)=$

$\sum_{m=0}^{M} q_m z^m$. Then, this last condition can be rewritten as

$$\sum_{m=0}^{M} E(q_m)(E(f))^m \neq 0.$$

Similarly, one can compute (8.1) at every $f \in \mathcal{R}$ such that $(I - E(fA))$ is invertible.

The arguments in [4, 2] are in the setting of power series (because one considers there the Hermite transform of the Kondratiev space rather than the Kondratiev space itself), and make use of derivatives. For the general case, when no power series are available, it is convenient to introduce the operators D_n, $n = 1, 2, \ldots$ defined by

$$D_n(\delta_\alpha) = \begin{cases} \alpha_n \delta_{\alpha - e_n} & \text{if } \alpha_n > 0 \\ 0 & \text{otherwise} \end{cases}$$

and by linear extension to any finite linear combination of such elements. We have in particular $D_n(XY) = D_n(X)Y + XD_n(Y)$.

As in [4] for $\mathcal{S}_{-1}$, given a Våge space $\mathcal{R}$ we define a rational function to be an expression of the form

$$D + zC(I - zA)^{-1}B,$$

where A, B, C and D are matrices of appropriate dimensions and with entries in $\mathcal{R}$. See Proposition 8.1 above. Before giving a sample result we recall that a pair $(C, A) \in \mathcal{R}^{p \times N} \times \mathcal{R}^{N \times N}$ is called *observable* if the map

$$f \mapsto \begin{pmatrix} Cf & CAf & CA^2 f & \cdots \end{pmatrix}$$

is injective from $\mathcal{R}^N$ into $(\mathcal{R}^p)^{\mathbb{N}}$. See [7, §2.2 p. 58]. In [4] it is proved for $\mathcal{S}_{-1}$ that an equivalent condition is:

$$C(I - zA)^{-1} f \equiv 0_{\mathcal{R}}^{p \times N} \Longrightarrow f = 0_{\mathcal{R}}^{N}.$$

The proof is the same for any Våge space.

Theorem 8.2. *Let $\hat{h}$ be a rational function with realization*

$$\hat{h}(z) = D + zC(I - zA)^{-1}B. \tag{8.2}$$

If the realization $E[\hat{h}](z) = E[D] + zE[C](I - zE[A])^{-1}E[B]$ is observable, then the realization (8.2) *is observable.*

Proof. We assume that $A \in \mathcal{R}^{N\times N}$. Let $f = \sum_{\alpha\in\ell_A} f_\alpha \delta_\alpha \in \mathcal{R}^N$ be such that $C(I-zA)^{-1}f \equiv 0$. We want to show that $f_\alpha = 0$ for all $\alpha \in \ell_A$. Since

$$E[\hat{h}](z) = E[D] + zE[C](I - zE[A])^{-1}E[B]$$

is an observable realization, we have that $E[C](I-zE[A])^{-1}E[f] \equiv 0$ implies $E[f] = 0$, and thus $f_0 = 0$. Now, since

$$\begin{aligned} D_n(C(I-zA)^{-1}f) &= D_n(C)(I-zA)^{-1}f + CD_n((I-zA)^{-1})f + \\ &\quad + C(I-zA)^{-1}D_n(f) \\ &= 0, \end{aligned}$$

and since $E[f] = 0$, we have that $E[C](I-zE[A])^{-1}E[D_n(f)] = 0$, and thus $f_{e_n} = 0$. Furthermore, by a simple induction, since

$$D_n^m(C(I-zA)^{-1}f) = \sum_{k<m} U_k D_n^k(f) + C(I-zA)^{-1}D_n^m(f)$$

for some U_k, and since $D_n^m(C(I-zA)^{-1}f) = 0$ we have that

$$E[C](I-zE[A])^{-1}E[D_n^m(f)] = 0,$$

and therefore $f_{me_n} = 0$. Thus, $f_\alpha = 0$ for all $\alpha \in \ell_A$ such that $\alpha = (0,\ldots,0,\alpha_n,0\ldots)$.
We may complete this proof as in [4]. □

In conclusion, Theorem 8.2 as well as Problem 4.9 (or more precisely, its solution presented in [2]) suggest that most of the classical linear system theory can be extended to our setting. This is important when one wants to take into account stochastic aspects of the theory. One such avenue consists of continuing the line of research initiated in [2, 3, 4]. One then considers input-output systems of the form

$$y_n = \sum_{k=0}^{n} h_k \diamond u_{n-k}, \quad n = 0, 1, \ldots$$

where the input sequence $(u_n)_{n\in\mathbb{N}_0}$ and the impulse response $(h_n)_{n\in\mathbb{N}_0}$ are in some Våge space. The choice of the given Våge space is done to express for instance that the system is stochastic (then one choses $\mathcal{S}_{-1}$). When the sequences consist of complex numbers, the Wick product reduces to the product of complex numbers, and we are back in the classical theory.

As already mentioned in Remark 6.8, another avenue is to define a stochastic linear system as a continuous mapping from the nuclear space $\mathscr{G} \otimes \mathcal{S}_{-1}$ into its dual, to use Schwartz's kernel theorem and then to follow Zemanian's approach to linear systems. See [30] for the latter.

Since the dual of $\mathscr{G} \otimes \mathcal{S}_{-1}$ is a Våge space, one can get more precise results than the ones in [30].

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